Analytic Multi-Solitonic Solutions of Variable-Coefficient Higher-Order Nonlinear Schrödinger Models by Modified Bilinear Method with Symbolic Computation

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In this paper, the physically interesting variable-coefficient higher-order nonlinear Schrödinger models in nonlinear optical fibers with varying higher-order effects such as third-order dispersion, self-steepening, delayed nonlinear response and gain or absorption are investigated. The bilinear transformation method is modified for constructing the analytic solutions of these models directly with sets of parametric conditions. With the aid of symbolic computation, the explicit analytic multi-solitonic solutions of the variable-coefficient higher-order nonlinear Schrödinger models are presented by employing the modified bilinear transformation method. The one- and two-solitonic solutions in explicit form are given in detail. Finally, solutions are illustrated and discussed through adjusting the parameters, so different dispersion management systems can be obtained.

Key words: Multi-Solitonic Solutions; Symbolic Computation; Variable-Coefficient Nonlinear Schrödinger Models; Modified Bilinear Method.

1. Introduction

The nonlinear Schrödinger (NLS)/perturbed NLS models with constant coefficients and/or variable coefficients are among the most important nonlinear models. Much attention has been paid to studying such models in many branches of modern science, such as plasmas physics [1 – 3], nonlinear modulation of waves in the Rayleigh-Taylor problem [4, 5], nonlinear pulse propagation in a long-distance, high speed optical fiber transmission system [6 – 11], and blood as an incompressible inviscid fluid by considering the arteries as a tapered elastic thin-walled long circularly conical tube [12]. With the development of the optical fiber transmission and soliton theory, the solitary light waves and optical solitons have been theoretically and experimentally of particular interest in the optical fiber system for the past decades [6, 7, 13 – 15]. The NLS type models are taken into account for investigating pulses in the picosecond regime. If the optical pulse is of the order of femtoseconds, the NLS type equations turn inadequate to describe the realistic problems. For this case, the higher-order effects such as third-order dispersion, self-steepening and delayed nonlinear response are considered; that is, the problem is governed by the higher-order nonlinear Schrödinger (HNLS) equation derived by Kodama and Hasegawa [16, 17]. The constant-coefficient HNLS model describing the ideal optical fiber transmission system has been studied such as by an improved algebraic method [18] and conducting the Painlevé analysis [19].

However, in the real fiber, when the core medium is inhomogeneous, the governing equation becomes the variable-coefficient HNLS (vc-HNLS) equation [16, 17, 20] as

$$\begin{aligned} &\mathrm{i}u_\xi + \alpha(\xi)u_{\tau\tau} + \beta(\xi)|u|^2u + \mathrm{i}\gamma(\xi)(|u|^2u)_\tau \\ &\quad + \mathrm{i}\zeta(\xi)(|u|^2)_\tau u + \mathrm{i}\delta(\xi)u_{\tau\tau\tau} = -\mathrm{i}\Gamma(\xi)u, \end{aligned} \tag{1}$$

where $u = u(\xi,\tau)$ is the complex envelope of the electrical field in a comoving frame, ξ the normalized propagation distance and τ the retarded time.

$\alpha(\xi)$ and $\delta(\xi)$ represent the group velocity dispersion, and third-order dispersion, respectively, and the third-order dispersion effect plays much more important roles in the femtosecond pulse compression [21]. $\beta(\xi)$ is the nonlinearity parameter, and the parameters $\gamma(\xi)$ and $\zeta(\xi)$ denote self-steepening and delayed nonlinear response effects, respectively. $\Gamma(\xi)$ is related to the heat-insulating amplification or absorption. All of these coefficient functions are real analytic ones. Equation (1) describes the femtosecond pulse propagation which can be applied extensively to telecommunication and ultrafast signal-routing systems in the weakly dispersive and nonlinear dielectrics with distributed parameters.

Equation (1) admits soliton-type solutions, called dispersion-managed solitons. The concept of dispersion management (DM) [22, 23] is a new and important development in the optical fiber transmission system. The goal of DM is to overcome some effects such as dispersive broadening of the pulse over long distances, resonant four-wave mixing and modulation instability. The utilization of the dispersion compensating fiber constitutes the main technique for DM. An optimal DM system can be devised owing to the appropriate distributed parameters in (1) for a concrete problem.

If the coefficient functions satisfy $\gamma(\xi)+\zeta(\xi)=0$, (1) turns to

$$\begin{aligned} &\mathrm{i}u_\xi+\alpha(\xi)u_{\tau\tau}+\beta(\xi)|u|^2u+\mathrm{i}\gamma(\xi)|u|^2u_\tau \\ &+\mathrm{i}\delta(\xi)u_{\tau\tau\tau}=-\mathrm{i}\Gamma(\xi)u. \end{aligned} \tag{2}$$

If the coefficients of (2) are constant, (2) is known as the Hirota equation for which the dark soliton [19] and N-envelope-soliton solutions [24] have been obtained. As for (2), when $\alpha(\xi)=\beta(\xi)=\Gamma(\xi)=0$ and $u(\xi,\tau)$ is a real function, the equation reduces to the variable-coefficient modified Korteweg-de Vries equation. Some other special cases of (1) in space or laboratory plasmas, fluid dynamics and optical fibers have been widely researched. For example, in the real communication system of optical solitons, pulse propagation is described by the equation

$$\mathrm{i}\psi_\xi+\beta(\xi)\psi_{\tau\tau}+\gamma(\xi)|\psi|^2\psi+\mathrm{i}g(\xi)\psi=0, \tag{3}$$

which can demonstrate the dispersion-managed unperturbed $\big(g(\xi)=0\big)$ [25–27] or perturbed $\big(g(\xi)\neq 0\big)$ [14, 15, 28, 29] variable-coefficient NLS models. As far as we know, the research on the vc-HNLS models is not yet widespread. Motivated by this, the vc-HNLS models are investigated in this paper.

The bilinear transformation method [30–32] is an analytical direct method for solving the N-soliton or multi-solitonic solutions of a wide class of nonlinear partial differential equations (NLPDEs) through dependent variable transformations and expansion of the format parameter [33]. It can also be used to derive the bilinear Bäcklund transformation [34, 35]. In fact, it is noted that the bilinear method has been focusing more on the constant-coefficient NLPDEs. In this paper, the modified bilinear method is introduced to construct the explicit analytic multi-solitonic solutions of the vc-HNLS models under certain parametric constraints. Due to the modified bilinear method, it is ready to transform the vc-HNLS models into the variable-coefficient bilinear equations similar to the corresponding constant-coefficient ones in form. Subsequently, families of analytic dispersion-managed solitonic solutions of the vc-HNLS models can be obtained.

This paper is organized as follows: In Section 2, the bilinear method is modified, and the bilinear form of (1) under certain parametric conditions and its one-solitonic solution are obtained. In Section 3, as a special case of (1), under the condition $\gamma(\xi)+\zeta(\xi)=0$, the N-solitonic solutions of (2) are constructed and an explicit two-solitonic solution is given in detail based on its bilinear form. In Section 4, there will be illustrations and discussions on those solutions.

2. Modified Bilinear Method and One-Solitonic Solution of (1)

The main features of the modified bilinear method, displayed by the modified dependent variable transformation, are given in the following. At the beginning, the nonlinear evolution equation is transformed into a bilinear form through the dependent variable transformation, i. e.,

$$u(\xi,\tau)=k(\xi)\frac{g(\xi,\tau)}{f(\xi,\tau)}, \tag{4}$$

where $k(\xi)$ and $f(\xi,\tau)$ are both real differentiable functions, and $g(\xi,\tau)$ is a complex differentiable one. Substituting transformation (4) into (1), and after some symbolic computations [36, 37], the following equation can be obtained:

$$\mathrm{i}\frac{[k(\xi)\Gamma(\xi)+k'(\xi)]}{f}g+\mathrm{i}\frac{k(\xi)}{f^2}D_\xi(g\cdot f)$$

$$
\begin{aligned}
&+\frac{\beta(\xi)k^3(\xi)}{f^3}g|g|^2+\frac{\alpha(\xi)k(\xi)}{f^2}D_\tau^2(g\cdot f)\\
&-\frac{\alpha(\xi)k(\xi)}{f^3}gD_\tau^2(f\cdot f)+\mathrm{i}\frac{3\gamma(\xi)k^3(\xi)}{f^4}D_\tau(g\cdot f)|g|^2\\
&+\mathrm{i}\frac{\gamma(\xi)k^3(\xi)}{f^3}gD_\tau(g^*\cdot g)+\mathrm{i}\frac{2\zeta(\xi)k^3(\xi)}{f^4}D_\tau(g\cdot f)|g|^2\\
&+\mathrm{i}\frac{\zeta(\xi)k^3(\xi)}{f^3}gD_\tau(g^*\cdot g)+\mathrm{i}\frac{\delta(\xi)k(\xi)}{f^2}D_\tau^3(g\cdot f)\\
&-\mathrm{i}\frac{3\delta(\xi)k(\xi)}{f^4}D_\tau(g\cdot f)D_\tau^2(f\cdot f)=0,
\end{aligned}\tag{5}
$$

where the prime and asterisk denote the derivative with respect to ξ and complex conjugate of the function $g(\xi,\tau)$, respectively, while D_ξ and D_τ are the bilinear derivative operators [30, 31] defined by

$$
D_\xi^m D_\tau^n(a\cdot b)=\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\xi'}\right)^m\left(\frac{\partial}{\partial\tau}-\frac{\partial}{\partial\tau'}\right)^n a(\xi,\tau)b(\xi',\tau')\Big|_{\xi'=\xi,\tau'=\tau}.\tag{6}
$$

Splitting (5) with the different powers of denominator f and considering the constraint on coefficients

$$
\frac{\beta(\xi)}{\alpha(\xi)}=\frac{3\gamma(\xi)+2\zeta(\xi)}{3\delta(\xi)},\tag{7}
$$

the resulting system of equations is derived:

$$
k(\xi)\Gamma(\xi)+k'(\xi)=0,\tag{8}
$$

$$
\begin{aligned}
&\frac{k(\xi)}{f^2}\left[\mathrm{i}D_\xi(g\cdot f)+\alpha(\xi)D_\tau^2(g\cdot f)+\mathrm{i}\delta(\xi)D_\tau^3(g\cdot f)\right]\\
&\quad+\mathrm{i}\frac{[\gamma(\xi)+\zeta(\xi)]k^3(\xi)}{f^4}gfD_\tau(g^*\cdot g)=0,
\end{aligned}\tag{9}
$$

$$
\begin{aligned}
&\frac{\alpha(\xi)k(\xi)}{f^4}gfD_\tau^2(f\cdot f)\\
&\quad+\mathrm{i}\frac{3\delta(\xi)k(\xi)}{f^4}D_\tau(g\cdot f)D_\tau^2(f\cdot f)\\
&\quad=\frac{\beta(\xi)k^3(\xi)}{f^4}gf|g|^2\\
&\qquad+\mathrm{i}\frac{[3\gamma(\xi)+2\zeta(\xi)]k^3(\xi)}{f^4}D_\tau(g\cdot f)|g|^2.
\end{aligned}\tag{10}
$$

According to (8), the relationship between $k(\xi)$ and $\Gamma(\xi)$ is proved to be

$$
k(\xi)=C_0\mathrm{e}^{-\int\Gamma(\xi)\mathrm{d}\xi},\tag{11}
$$

where C_0 is a nonzero real constant of integration for nontrivial solutions of (1). Symbolic computation on (9) and (10) under the constraint (7) leads to

$$
\alpha(\xi)D_\tau^2(f\cdot f)=\beta(\xi)k^2(\xi)|g|^2,\tag{12}
$$

$$
\begin{aligned}
&f^2\left[\mathrm{i}D_\xi+\alpha(\xi)D_\tau^2+\mathrm{i}\delta(\xi)D_\tau^3\right](g\cdot f)\\
&\quad+\mathrm{i}\left[\gamma(\xi)+\zeta(\xi)\right]k^2(\xi)gfD_\tau(g^*\cdot g)=0.
\end{aligned}\tag{13}
$$

That is to say, under conditions (7) and (11), the bilinear form of (1) consisting of (12) and (13) is obtained. The following will be to solve the bilinear form and get the one-solitonic solution of (1). Let us expand $f(\xi,\tau)$ and $g(\xi,\tau)$ formally in powers of an arbitrary parameter ε as

$$
\begin{aligned}
f(\xi,\tau)=1&+\varepsilon f_1(\xi,\tau)+\varepsilon^2 f_2(\xi,\tau)\\
&+\varepsilon^3 f_3(\xi,\tau)+\cdots,
\end{aligned}\tag{14}
$$

$$
\begin{aligned}
g(\xi,\tau)=\,&\varepsilon g_1(\xi,\tau)+\varepsilon^2 g_2(\xi,\tau)\\
&+\varepsilon^3 g_3(\xi,\tau)+\cdots,
\end{aligned}\tag{15}
$$

and note

$$
\Psi=\mathrm{i}D_\xi+\alpha(\xi)D_\tau^2+\mathrm{i}\delta(\xi)D_\tau^3
$$

in (13). Substituting expressions (14) and (15) into (12) and (13) and equating coefficients of like powers of ε to zero yield the recursion relations for $f_n(\xi,\tau)$ and $g_n(\xi,\tau)$ $(n=0,1,2,\cdots)$:

$$
\begin{aligned}
2\alpha(\xi)D_\tau^2(f_n\cdot 1)=&-\alpha(\xi)D_\tau^2\left(\sum_{k=1}^{n-1}f_k\cdot f_{n-k}\right)\\
&+\beta(\xi)k^2(\xi)\left(\sum_{k=0}^{n-1}g_k^*\cdot g_{n-k}\right),
\end{aligned}\tag{16}
$$

$$
\begin{aligned}
&\Psi(g_n\cdot 1)=-\Psi\left(\sum_{k=1}^{n-1}g_k\cdot f_{n-k}\right)\\
&-\sum_{k=1}^{n-1}\left[\left(\sum_{j=0}^{k}f_j\cdot f_{k-j}\right)\cdot\Psi\left(\sum_{j=1}^{n-k}g_j\cdot f_{n-k-j}\right)\right]\\
&-\mathrm{i}[\gamma(\xi)+\zeta(\xi)]k^2(\xi)\\
&\cdot\sum_{k=1}^{n-2}\left[\left(\sum_{j=1}^{k}g_j\cdot f_{k-j}\right)\cdot D_\tau\left(\sum_{j=1}^{n-k-1}g_j^*\cdot g_{n-k-j}\right)\right],
\end{aligned}\tag{17}
$$

noting $f_0(\xi,\tau)=1$ and $g_0(\xi,\tau)=0$. For $n=0$,

$$
\alpha(\xi)D_\tau^2(1\cdot 1)=0
$$

is satisfied identically. In the following, the one-solitonic solution of (1) will be searched out.

To obtain the one-solitonic solution of (1), without loss of generality, setting $\varepsilon = 1$ and assuming that

$$f(\xi,\tau) = 1 + A\mathrm{e}^{\theta+\theta^*}, \quad g(\xi,\tau) = \mathrm{e}^{\theta},$$

where A is a real constant to be determined and $\theta = \eta\tau + \varphi(\xi)$ with η as a real constant and $\varphi(\xi)$ as a differentiable function to be determined. Substituting the supposition of $f(\xi,\tau)$ and $g(\xi,\tau)$ into (12) and (13), and after some calculations, $\varphi(\xi)$ is determined to be

$$\varphi(\xi) = \mathrm{i}\eta^2 \int \alpha(\xi)\mathrm{d}\xi - \eta^3 \int \delta(\xi)\mathrm{d}\xi + \varphi_0,$$

where φ_0 is a real constant of integration. Further, the relation

$$\frac{\beta(\xi)}{\alpha(\xi)} = C_1 \mathrm{e}^{2\int \Gamma(\xi)\mathrm{d}\xi} \tag{18}$$

is presented with C_1 as a real integral constant. Hereby A turns out to be

$$A = \frac{C_0^2 C_1}{8\eta^2}.$$

Consequently, $f(\xi,\tau)$ and $g(\xi,\tau)$ can be reduced to

$$f(\xi,\tau) = 1 + \frac{C_0^2 C_1}{8\eta^2}\mathrm{e}^{\theta+\theta^*}, \quad g(\xi,t) = \mathrm{e}^{\theta},$$

with $\theta = \eta\tau + \mathrm{i}\eta^2\int\alpha(\xi)\mathrm{d}\xi - \eta^3\int\delta(\xi)\mathrm{d}\xi + \varphi_0$.

To sum up, under the parametric constraints (7), (10) and (18), the exact DM one-solitonic solution of (1) in explicit form yields

$$\begin{aligned} u(\xi,\tau) &= C_0 \mathrm{e}^{-\int\Gamma(\xi)\mathrm{d}\xi} \frac{\mathrm{e}^{\eta\tau + \mathrm{i}\eta^2\int\alpha(\xi)\mathrm{d}\xi - \eta^3\int\delta(\xi)\mathrm{d}\xi + \varphi_0}}{1 + \frac{C_0^2 C_1}{8\eta^2}\mathrm{e}^{2(\eta\tau - \eta^3\int\delta(\xi)\mathrm{d}\xi + \varphi_0)}} \\ &= \sqrt{\frac{2}{C_1}}\eta \mathrm{e}^{\mathrm{i}\eta^2\int\alpha(\xi)\mathrm{d}\xi - \int\Gamma(\xi)\mathrm{d}\xi} \\ &\quad \cdot \operatorname{sech}[\eta\tau - \eta^3 \int\delta(\xi)\mathrm{d}\xi + \sigma], \end{aligned} \tag{19}$$

where $\sigma = \frac{1}{2}\ln\left(\frac{C_0^2 C_1}{8\eta^2}\right) + \varphi_0$.

3. Multi-Solitonic Solutions of (2)

When the coefficients $\gamma(\xi)$ and $\zeta(\xi)$ satisfy

$$\gamma(\xi) + \zeta(\xi) = 0, \tag{20}$$

(1) becomes (2), and condition (7) turns to

$$\frac{\beta(\xi)}{\alpha(\xi)} = \frac{\gamma(\xi)}{3\delta(\xi)}, \tag{21}$$

which can be regarded as the generalized Hirota condition [20, 38].

In this section, the explicit two-solitonic and N-solitonic solutions of (2) will be gained under conditions (11), (18) and (21).

At first, the bilinear form (12) and (13) become

$$\alpha(\xi) D_\tau^2 (f\cdot f) = \beta(\xi) k^2(\xi) |g|^2, \tag{22}$$

$$\Psi(g\cdot f) = 0, \tag{23}$$

under conditions (11), (18) and (21). Substituting expressions (14) and (15) into (22) and (23), the recursive relations of $f_n(\xi,\tau)$ and $g_n(\xi,\tau)$ $(n = 1, 2, \cdots)$ can be shown:

$$\begin{aligned} 2\alpha(\xi) D_\tau^2 (f_n \cdot 1) = &-\alpha(\xi) D_\tau^2 \left(\sum_{k=1}^{n-1} f_k \cdot f_{n-k} \right) \\ &+ \beta(\xi) k^2(\xi) \left(\sum_{k=0}^{n} g_k^* \cdot g_{n-k} \right), \end{aligned} \tag{24}$$

$$\Psi(g_n \cdot 1) = -\Psi\left(\sum_{k=1}^{n-1} g_k \cdot f_{n-k} \right). \tag{25}$$

To generate the two-solitonic solution, the following ansatz is made:

$$\begin{aligned} f(\xi,\tau) = 1 &+ A_1 \mathrm{e}^{\theta_1+\theta_1^*} + A_2 \mathrm{e}^{\theta_2+\theta_2^*} + A_3 \mathrm{e}^{\theta_2+\theta_1^*} \\ &+ A_4 \mathrm{e}^{\theta_1+\theta_2^*} + A_5 \mathrm{e}^{\theta_1+\theta_2+\theta_1^*+\theta_2^*}, \end{aligned}$$

$$g(\xi,\tau) = \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + A_6 \mathrm{e}^{\theta_1+\theta_2+\theta_1^*} + A_7 \mathrm{e}^{\theta_1+\theta_2+\theta_2^*},$$

where $\theta_j = \eta_j\tau + \mathrm{i}\eta_j^2\int\alpha(\xi)\mathrm{d}\xi - \eta_j^3\int\delta(\xi)\mathrm{d}\xi + \varphi_{0j}$ with η_j and φ_{0j} $(j = 1, 2)$ as real constants, and A_j $(j = 1, \cdots, 7)$ as real constants to be determined. According to the recursive relations, A_j $(j = 1, \cdots, 7)$ are found to be

$$A_1 = \frac{C_0^2 C_1}{8\eta_1^2}, \quad A_2 = \frac{C_0^2 C_1}{8\eta_2^2}, \quad A_3 = A_4 = \frac{C_0^2 C_1}{2(\eta_1+\eta_2)^2},$$

$$A_5 = \frac{C_0^4 C_1^2 (\eta_1-\eta_2)^4}{64\eta_1^2\eta_2^2(\eta_1+\eta_2)^4}, \quad A_6 = \frac{C_0^2 C_1 (\eta_1-\eta_2)^2}{8\eta_1^2(\eta_1+\eta_2)^2},$$

$$A_7 = \frac{C_0^2 C_1 (\eta_1-\eta_2)^2}{8\eta_2^2(\eta_1+\eta_2)^2}.$$

Thus, $f(\xi,\tau)$ and $g(\xi,\tau)$ are given as

$$f(\xi,\tau) = 1 + \frac{C_0^2 C_1}{8\eta_1^2} e^{\theta_1+\theta_1^*} + \frac{C_0^2 C_1}{8\eta_2^2} e^{\theta_2+\theta_2^*} + \frac{C_0^2 C_1}{2(\eta_1+\eta_2)^2} e^{\theta_2+\theta_1^*} + \frac{C_0^2 C_1}{2(\eta_1+\eta_2)^2} e^{\theta_1+\theta_2^*} + \frac{C_0^4 C_1^2 (\eta_1-\eta_2)^4}{64\eta_1^2\eta_2^2(\eta_1+\eta_2)^4} e^{\theta_1+\theta_2+\theta_1^*+\theta_2^*},$$

$$g(\xi,\tau) = e^{\theta_1} + e^{\theta_2} + \frac{C_0^2 C_1 (\eta_1-\eta_2)^2}{8\eta_1^2(\eta_1+\eta_2)^2} e^{\theta_1+\theta_2+\theta_1^*} + \frac{C_0^2 C_1 (\eta_1-\eta_2)^2}{8\eta_2^2(\eta_1+\eta_2)^2} e^{\theta_1+\theta_2+\theta_2^*}.$$

Therefore, the DM two-solitonic solution of (2) under conditions (11), (18) and (21) are sought in the form

$$u(\xi,\tau) = k(\xi)\frac{g(\xi,\tau)}{f(\xi,\tau)} = \sqrt{\frac{8}{C_0^2 C_1}} \cdot \frac{\eta_1+\eta_2}{\eta_1-\eta_2} e^{-\int\Gamma(\xi)d\xi} \left\{ \eta_1 \cosh(\varepsilon_1+\sigma_1) e^{i\eta_2^2\int\alpha(\xi)d\xi} + \eta_2 \cosh(\varepsilon_2+\sigma_2) e^{i\eta_1^2\int\alpha(\xi)d\xi} \right\} \cdot \left\{ \cosh(\varepsilon_1+\varepsilon_2+\sigma_3) + \frac{(\eta_1+\eta_2)^2}{(\eta_1-\eta_2)^2}\cosh(\varepsilon_2-\varepsilon_1+\sigma_4) + \frac{8\eta_1\eta_2}{(\eta_1-\eta_2)^2}\cos\left[(\eta_1^2-\eta_2^2)\int\alpha(\xi)d\xi\right] \right\}^{-1}. \quad (26)$$

with

$$\varepsilon_j = \eta_j\tau - \eta_j^3\int\delta(\xi)d\xi + \varphi_{0j} \quad (j=1,2),$$

$$\sigma_j = \ln\left(\sqrt{\frac{C_0^2 C_1(\eta_1-\eta_2)^2}{8\eta_j^2(\eta_1+\eta_2)^2}}\right) \quad (j=1,2),$$

$$\sigma_3 = \ln\left(\frac{C_0^2 C_1(\eta_1-\eta_2)^2}{8\eta_1\eta_2(\eta_1+\eta_2)^2}\right),$$

$$\sigma_4 = \ln\left(\frac{\eta_1}{\eta_2}\right).$$

The N-solitonic solution in the sense of [24] of (2) can be expressed in the form

$$u(\xi,\tau) = C_0 e^{-\int\Gamma(\xi)d\xi}\frac{g(\xi,\tau)}{f(\xi,\tau)}, \quad (27)$$

$$f(\xi,\tau) = \sum_{\mu=0,1}{}' \exp\left(\sum_{j<k}^{2N} B_{jk}\mu_j\mu_k + \sum_{j=1}^{2N}\mu_j\theta_j\right), \quad (28)$$

$$g(\xi,\tau) = \sum_{\mu=0,1}{}'' \exp\left(\sum_{j<k}^{2N} B_{jk}\mu_j\mu_k + \sum_{j=1}^{2N}\mu_j\theta_j\right), \quad (29)$$

$$g^*(\xi,\tau) = \sum_{\mu=0,1}{}''' \exp\left(\sum_{j<k}^{2N} B_{jk}\mu_j\mu_k + \sum_{j=1}^{2N}\mu_j\theta_j\right), \quad (30)$$

where

$$\theta_j = \eta_j\tau + i\eta_j^2\int\alpha(\xi)d\xi - \eta_j^3\int\delta(\xi)d\xi + \varphi_{0j} \quad (j=1,2,\cdots,2N),$$

$$\omega_j = i\eta_j^2\int\alpha(\xi)d\xi - \eta_j^3\int\delta(\xi)d\xi + \varphi_{0j} \quad (j=1,2,\cdots,2N),$$

$$\eta_{j+N} = \eta_j, \quad \omega_{j+N} = \omega_j^* \quad (j=1,2,\cdots,N),$$

$$B_{jk} = \ln\left[\frac{2\alpha(\xi)}{\beta(\xi)k^2(\xi)}(\eta_j-\eta_k)^2\right]$$

for $j=1,2,\cdots,N$ and $k=1,2,\cdots,N$, or $j=N+1,\cdots,2N$ and $k=N+1,\cdots,2N$,

$$B_{jk} = -\ln\left[\frac{2\alpha(\xi)}{\beta(\xi)k^2(\xi)}(\eta_j+\eta_k)^2\right]$$

for $j=1,2,\cdots,N$ and $k=N+1,\cdots,2N$,

with η_j and φ_{0j} as real constants ($j=1,2,\cdots,2N$) and $i=\sqrt{-1}$. $\sum_{\mu=0,1}{}'$, $\sum_{\mu=0,1}{}''$ and $\sum_{\mu=0,1}{}'''$ denote the summation over all possible combinations of $\mu_j = 0,1$ ($j=1,2,\cdots,2N$) and require

$$\sum_{j=1}^{N}{}'\mu_j = \sum_{j=1}^{N}{}'\mu_{j+N},$$

$$\sum_{j=1}^{N}{}''\mu_j = 1 + \sum_{j=1}^{N}{}''\mu_{j+N},$$

$$1 + \sum_{j=1}^{N}{}'''\mu_j = \sum_{j=1}^{N}{}'''\mu_{j+N}.$$

4. Discussion and Conclusion

The optical solitons are of particular interest theoretically and experimentally because of their potential applications in the long-distance, high speed optical fiber transmission system. The vc-HNLS models describe the femtosecond pulse propagation which is applied to telecommunication and ultrafast signal-routing

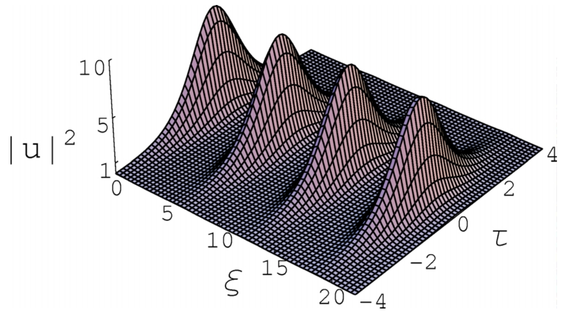

Fig. 1. DM bright one-solitonic solution with $C_0 = 1$, $C_1 = 1$, $\eta = 0.8$, $\sigma = 0$, $\Gamma(\xi) = \sin(\xi)$ and $\delta(\xi) = 0.1\sin(\xi)$.

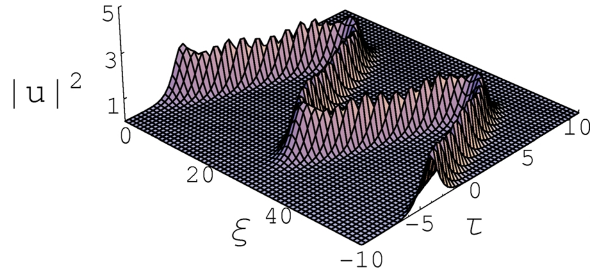

Fig. 3. DM one-solitonic solution demonstrating oscillation with the same parameters as in Fig. 1 except that $\Gamma(\xi) = 0.2\sin(5\xi)$ and $\delta(\xi) = \sin(\xi/5)$.

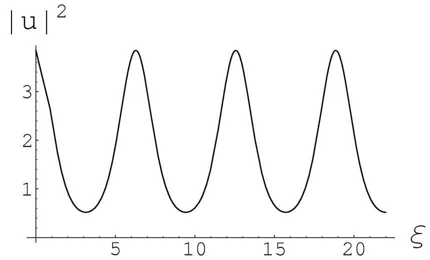

Fig. 2. Trajectory of the peak of the DM one-solitonic solution, i. e., at the moment of $\tau = -\cos(\xi)$ with the same parameters as in Figure 1.

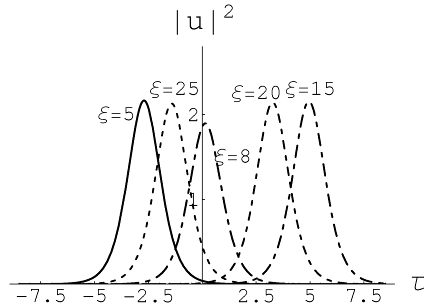

Fig. 4. Oscillation of the peak for the DM one-solitonic solution indicated with the locations at $\xi = 5$, $\xi = 8$, $\xi = 15$, $\xi = 20$, $\xi = 25$.

systems extensively in the weakly dispersive and nonlinear dielectrics with distributed parameters. In this paper, according to the modified bilinear method, one-solitonic solution of (1) is derived under conditions (7), (11) and (18). *N*-Solitonic and two-solitonic solutions of (2) in explicit form are also presented under constraints (11), (18), and (21). We believe that the modified bilinear method similar to the bilinear method, which is one of the effective methods for constructing solitonic solutions, can be applied to obtain multi-solitonic solutions of other variable-coefficient nonlinear evolution equations.

In the following, discussions and conclusions on the solutions will be demonstrated.

1. Through the direct but powerful modified bilinear method, under certain coefficient conditions, an explicit one-solitonic solution of (1) has been obtained. The weak balance among third-order dispersion, self-steeping and delayed nonlinear response effects, i. e., condition (7), induces the existence of the one-solitonic solution. Moreover, the third-order dispersion effect affects the propagation velocity of the soliton, and the phase shift is related to the group velocity dispersion, while the perturbed term $\Gamma(\xi)$ has influence on the the wave amplitude. In view of the form of solution and constraint (18), adjusting the parameters $\alpha(\xi)$, $\beta(\xi)$ and $\delta(\xi)$ can lead to an optimal dispersion management system for the concrete problem. With symbolic computation, it is found that solution (19) is also the one-solitonic solution of (2) as a special case of (1).

2. For the vc-HNLS models (1) and (2) with variable coefficients depending on the normalized propagation distance ξ, the amplitude of the one-solitonic solution goes through periodic growth and delay, with the parameters as seen in Figure 1. The periodicity of the peak of the one-solitonic solution can be observed in Figure 2.

3. Figures 3 and 4 show the oscillation of the peak of the DM one-solitonic solution. With the pulse propagation along the distance ξ, the peak of the one-solitonic solution demonstrates oscillation along the time axis τ. Meanwhile, for the parameter $\Gamma(\xi)$ adopted in Fig. 3, the amplitude goes periodic changes along the distance.

4. Adjusting the parameters accordingly, different DM soliton systems can be obtained. Figures 5 and 6 demonstrate the DM two-solitonic interactions. From Fig. 5, it can be noted that two solitons collide elas-

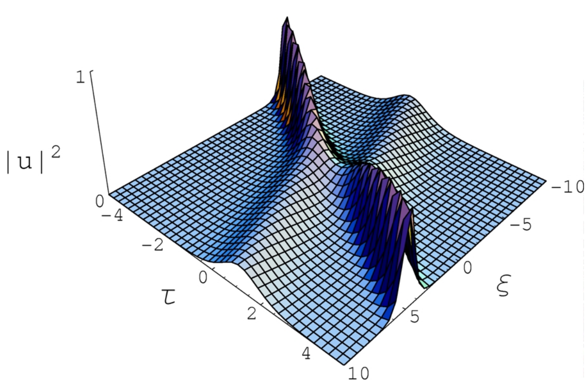

Fig. 5. DM two-solitonic solution interaction with $C_0 = 2$, $C_1 = 10$, $\eta_1 = 1$, $\eta_2 = 2$, $\varphi_{01} = \varphi_{02} = 2$, $\alpha(\xi) = 2$, $\Gamma(\xi) = 0$ and $\delta(\xi) = 0.2$.

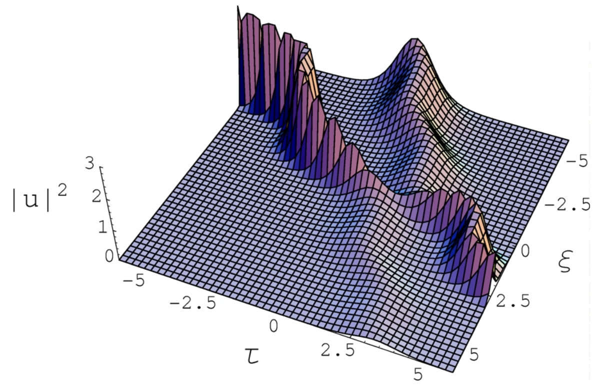

Fig. 6. DM two-solitonic solution interaction with the same parameters as in Fig. 5 except that $C_0 = 1$, $C_1 = 1$, $\Gamma(\xi) = 0.1$, and $\delta(\xi) = \cos^2(\xi)$.

tically without perturbation, i. e., after collision the two-solitonic waves maintain their original shapes with only a phase shift at the moment of collision. However, for Fig. 6 the perturbation $\Gamma(\xi)$ causes the amplitude attenuation and destroys the elasticity of soliton collision [39]. The two cases have in common that the taller wave catches the shorter one before collision, and leaves the shorter wave behind after collision.

In conclusion, the higher-order nonlinear Schrödinger models govern the propagation of femtosecond light pulses in the optical fibers. The femtosecond soliton control is of physical interest while the picosecond soliton control has been extensively studied. In this paper, the explicit DM multi-solitonic solutions of the vc-HNLS models are obtained through the direct modified bilinear method under certain parametric conditions. From the optical point of view, those conditions may be of guidance for the experimental generation of solitons. Evidently, the modified bilinear method can also be applied to obtain multi-solitonic solutions of families of variable-coefficient NLPDEs in the sense of [24].

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